

Asymptotic behavior at isolated singularities for solutions of nonlocal semilinear elliptic systems of inequalities

Marius Ghergu* and Steven D. Taliaferro^{†‡}

Abstract

We study the behavior near the origin of C^2 positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{cases} 0 \leq -\Delta u \leq \left(\frac{1}{|x|^\alpha} * v \right)^\lambda \\ 0 \leq -\Delta v \leq \left(\frac{1}{|x|^\beta} * u \right)^\sigma \end{cases} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3,$$

where $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (0, n)$.

A by-product of our methods used to study these solutions will be results on the behavior near the origin of $L^1(B_1(0))$ solutions f and g of the system

$$\begin{cases} 0 \leq f(x) \leq C \left(|x|^{2-\alpha} + \int_{|y|<1} \frac{g(y) dy}{|x-y|^{\alpha-2}} \right)^\lambda \\ 0 \leq g(x) \leq C \left(|x|^{2-\beta} + \int_{|y|<1} \frac{f(y) dy}{|x-y|^{\beta-2}} \right)^\sigma \end{cases} \quad \text{for } 0 < |x| < 1$$

where $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (2, n+2)$.

2010 Mathematics Subject Classification. 35B09, 35B33, 35B40, 35J47, 35J60, 35J91, 35R45.

Contents

1	Introduction	2
2	Statement of the main results	4
2.1	Results for system (1.1)	4
2.2	Results for system (1.6)	8
3	Preliminary results	10
3.1	Nonlinear Riesz potentials	10
3.2	Further estimates	11
4	Proof of Theorem 2.1	15
5	Proof of Theorem 2.2	16

*School of Mathematical Sciences, University College Dublin, Belfield, Dublin 4, Ireland; marius.ghergu@ucd.ie

[†]Mathematics Department, Texas A&M University, College Station, TX 77843-3368; stalia@math.tamu.edu

[‡]Corresponding author, Phone 001-979-845-3261, Fax 001-979-845-6028

6	Proof of Theorems 2.3–2.5, 2.9, and 2.10	17
7	Proof of Theorem 2.6	24
8	Proof of Theorem 2.7	25
9	Proof of Theorem 2.11	27
10	Proof of Theorem 2.12	29

1 Introduction

In this paper we study the behavior near the origin of $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions $u(x)$ and $v(x)$ of the system

$$\begin{cases} 0 \leq -\Delta u \leq \left(\frac{1}{|x|^\alpha} * v \right)^\lambda \\ 0 \leq -\Delta v \leq \left(\frac{1}{|x|^\beta} * u \right)^\sigma \end{cases} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, \quad (1.1)$$

where $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (0, n)$.

The goal of this work is to address the following question.

Question 1. For which constants $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (0, n)$ do there exist continuous functions $h_1, h_2 : (0, 1) \rightarrow (0, \infty)$ such that all $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions $u(x)$ and $v(x)$ of the system (1.1) satisfy

$$\begin{aligned} u(x) &= \mathcal{O}(h_1(|x|)) \quad \text{as } x \rightarrow 0 \\ v(x) &= \mathcal{O}(h_2(|x|)) \quad \text{as } x \rightarrow 0 \end{aligned}$$

and what are the optimal such h_1 and h_2 when they exist?

We call a function h_1 (resp. h_2) with the above properties a *pointwise bound* for u (resp. v) as $x \rightarrow 0$.

Remark 1. Let $\Gamma \in C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ be a positive function such that $\Gamma(x) = |x|^{-(n-2)}$ for $0 < |x| < 2$. Since $-\Delta \Gamma = 0$ in $B_2(0) \setminus \{0\}$, the functions $u_0(x) = v_0(x) = \Gamma(x)$ are always positive solutions of (1.1). Hence, any pointwise bound for positive solutions of (1.1) must be at least as large as $|x|^{-(n-2)}$ and whenever $|x|^{-(n-2)}$ is such a bound for u (resp. v) it is necessarily optimal. In this case we say that u (resp. v) is *harmonically bounded* at 0.

A first motivation for the study of (1.1) comes from the equation

$$-\Delta u = \left(\frac{1}{|x|^\alpha} * u^p \right) |u|^{p-2} u \quad \text{in } \mathbb{R}^n, \quad (1.2)$$

where $\alpha \in (0, n)$ and $p > 1$. For $n = 3$ and $\alpha = p = 2$, equation (1.2) is known in the literature as the *Choquard-Pekar equation* and was introduced in [19] as a model in quantum theory of a Polaron at rest (see also [6]). Later, the equation (1.2) appears as a model of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [17]. More recently, the same equation (1.2) was used in a model of self-gravitating matter (see, e.g., [13, 18]) and it is

known in this context as the *Schrödinger-Newton equation*. In the degenerate case $p = 1$, equation (1.2) becomes the prototype for our system (1.1).

Another motivation for the study of (1.1) is given by various integral equations that have been recently investigated. For instance, the system

$$\begin{cases} u(x) = \left(\int_{\mathbb{R}^n} \frac{v(y)}{|x-y|^\alpha} dy \right)^\lambda \\ v(x) = \left(\int_{\mathbb{R}^n} \frac{u(y)}{|x-y|^\beta} dy \right)^\sigma \end{cases} \quad \text{in } \mathbb{R}^n, n \geq 3, \quad (1.3)$$

and its more general forms appear in [4, 5, 12, 14, 15, 16]. These works are mainly concerned with radial symmetry, monotonicity or regularity of solutions.

As emphasized in [12, 16], the system (1.3) is related to the Hardy-Littlewood-Sobolev inequality

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\theta} dx dy \right| \leq C(n) \|f\|_p \|g\|_q, \quad (1.4)$$

where $p, q > 1$ and $\theta = (2 - 1/p - 1/q)n$.

In order to find the best constant in (1.4) one has to find

$$J := \min_{\|f\|_p=1, \|g\|_q=1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^\theta} dx dy$$

and this leads to the Euler-Lagrange equations

$$\begin{cases} f(x) = \left(\frac{1}{J} \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^\theta} dy \right)^\lambda \\ g(x) = \left(\frac{1}{J} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^\theta} dy \right)^\sigma \end{cases} \quad \text{in } \mathbb{R}^n, \quad (1.5)$$

where $\lambda = 1/(p-1) > 0$ and $\sigma = 1/(q-1) > 0$.

We point out that (1.1) has a similar structure to (1.5). Indeed, by the well known result of Brezis and Lions [2] (see Lemma 3.2 below) regarding the representation of nonnegative superharmonic functions in the punctured ball, positive solutions u and v of (1.1) satisfy

$$-\Delta u, -\Delta v \in L^1(B_1(0))$$

and

$$\begin{cases} u(x) \leq C \left(|x|^{2-n} + \int_{|y|<1} \frac{-\Delta u(y) dy}{|x-y|^{n-2}} \right) \\ v(x) \leq C \left(|x|^{2-n} + \int_{|y|<1} \frac{-\Delta v(y) dy}{|x-y|^{n-2}} \right) \end{cases} \quad \text{for } 0 < |x| < 1.$$

Substituting these estimates in (1.1) and using Lemma 3.9 and Corollary 3.7 below, we find for $\alpha, \beta \in (2, n)$ that

$$\begin{cases} 0 \leq f(x) \leq M \left(|x|^{2-\alpha} + \int_{|y|<1} \frac{g(y) dy}{|x-y|^{\alpha-2}} \right)^\lambda \\ 0 \leq g(x) \leq M \left(|x|^{2-\beta} + \int_{|y|<1} \frac{f(y) dy}{|x-y|^{\beta-2}} \right)^\sigma \end{cases} \quad \text{for } 0 < |x| < 1, \quad (1.6)$$

where $f = -\Delta u$, $g = -\Delta v$ are $C(\mathbb{R}^n \setminus \{0\}) \cap L^1(B_1(0))$ functions and M is a positive constant.

A by-product of our methods used to study solutions of (1.1) will be results on the behavior near the origin of $L^1(B_1(0))$ solutions f and g of (1.6) when $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (2, n+2)$.

Before we state the main results for (1.1) let us mention the following system which we considered in [8]:

$$\begin{cases} 0 \leq -\Delta u \leq v^\lambda \\ 0 \leq -\Delta v \leq u^\sigma \end{cases} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, \quad (1.7)$$

where $\lambda, \sigma \geq 0$. In [8] we emphasized the existence of a critical curve in the $\lambda\sigma$ -plane that optimally describes the existence of pointwise bounds for (1.7). A particular feature of (1.7) is that whenever pointwise bounds exist, then at least one of u and v must be harmonically bounded. We shall see that this is not always the case when dealing with the nonlocal system (1.1). Theorems 2.5 and 2.6 below illustrate such a phenomenon which we believe is due to the more complex character of (1.1) that involves four parameters $\alpha, \beta, \lambda, \sigma$ (instead of two parameters in the case of (1.7)).

Since positive solutions u and v of the system of inequalities (1.1) (resp. (1.7)) are also solutions of the system of equations

$$\begin{cases} -\Delta u = \left(\frac{1}{|x|^\alpha} * v \right)^\lambda \\ -\Delta v = \left(\frac{1}{|x|^\beta} * u \right)^\sigma \end{cases} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, \quad (1.8)$$

$$\left(\text{resp. } \begin{cases} -\Delta u = v^\lambda \\ -\Delta v = u^\sigma \end{cases} \quad \text{in } B_1(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3, \right) \quad (1.9)$$

our pointwise bounds at 0 for solutions of the systems (1.1) and (1.7) also hold for solutions of the systems (1.8) and (1.9) respectively. Such bounds are often a first step for obtaining more precise asymptotic behavior at 0 of positive solutions of systems (1.8) and (1.9) and nonexistence of entire solutions. The system (1.9) has been studied extensively. See for example [1] and [21].

2 Statement of the main results

2.1 Results for system (1.1)

We first consider the case that either α or β belongs to the interval $(0, 2]$. We can assume without loss of generality that $\beta \in (0, 2]$.

Theorem 2.1. *Suppose*

$$\alpha \in (0, n), \quad \beta \in (0, 2], \quad \text{and} \quad \lambda, \sigma \geq 0.$$

Let u and v be $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions of (1.1). Then

$$u(x) = \begin{cases} \mathcal{O}(|x|^{-(n-2)}) & \text{if } n \geq \lambda(\alpha - 2), \\ o\left(|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) & \text{if } n < \lambda(\alpha - 2), \end{cases} \quad \text{as } x \rightarrow 0, \quad (2.1)$$

and

$$v(x) = \mathcal{O}(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0. \quad (2.2)$$

By Remark 1 the estimate (2.2) and the first estimate in (2.1) are optimal. By the following theorem, the second estimate in (2.1) is also optimal.

Theorem 2.2. *Suppose*

$$0 < \beta \leq 2 < \alpha < n \quad \text{and} \quad \lambda > \frac{n}{\alpha - 2}.$$

Let $h : (0, 1) \rightarrow (0, 1)$ be a continuous function satisfying $\lim_{t \rightarrow 0^+} h(t) = 0$. Then there exist $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions u and v of (1.1) such that

$$u(x) \neq \mathcal{O}\left(h(|x|)|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0 \quad (2.3)$$

and

$$v(x)|x|^{n-2} \rightarrow 1 \quad \text{as } x \rightarrow 0. \quad (2.4)$$

Note that, according to Theorem 2.1, if $\alpha, \beta \in (0, 2]$ then all positive solutions u and v of (1.1) are harmonically bounded, that is

$$u(x) = \mathcal{O}(|x|^{-(n-2)}) \quad \text{and} \quad v(x) = \mathcal{O}(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0, \quad (2.5)$$

regardless of the size of the exponents λ and σ .

We next consider the case that $\alpha, \beta \in (2, n)$. In this setting the study of the asymptotic behavior is more delicate and it involves all parameters α, β, λ and σ . We can assume without loss of generality that

$$0 \leq (\beta - 2)\sigma \leq (\alpha - 2)\lambda. \quad (2.6)$$

Let $\alpha, \beta \in (2, n)$ be fixed constants. If λ and σ satisfy (2.6) then (λ, σ) belongs to one of the following five pairwise disjoint subsets of the $\lambda\sigma$ -plane.

$$\begin{aligned} A &:= \left\{ (\lambda, \sigma) : 0 \leq \lambda \leq \frac{n}{\alpha - 2} \quad \text{and} \quad 0 \leq \sigma \leq \frac{\alpha - 2}{\beta - 2}\lambda \right\} \setminus \left\{ \left(\frac{n}{\alpha - 2}, \frac{n}{\beta - 2} \right) \right\} \\ B &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad 0 \leq \sigma \leq \frac{2}{\beta - 2} + \frac{n(n-2)}{(\alpha - 2)(\beta - 2)} \frac{1}{\lambda} \right\} \\ C &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad \frac{2}{\beta - 2} + \frac{n(n-2)}{(\alpha - 2)(\beta - 2)} \frac{1}{\lambda} < \sigma < \frac{n+2-\alpha}{\beta - 2} + \frac{n}{\beta - 2} \frac{1}{\lambda} \right\} \\ D &:= \left\{ (\lambda, \sigma) : \lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad \frac{n+2-\alpha}{\beta - 2} + \frac{n}{\beta - 2} \frac{1}{\lambda} < \sigma \leq \frac{\alpha - 2}{\beta - 2}\lambda \right\} \\ E &:= \left\{ (\lambda, \sigma) : \lambda \geq \frac{n}{\alpha - 2} \quad \text{and} \quad \sigma = \frac{n+2-\alpha}{\beta - 2} + \frac{n}{\beta - 2} \frac{1}{\lambda} \right\}. \end{aligned}$$

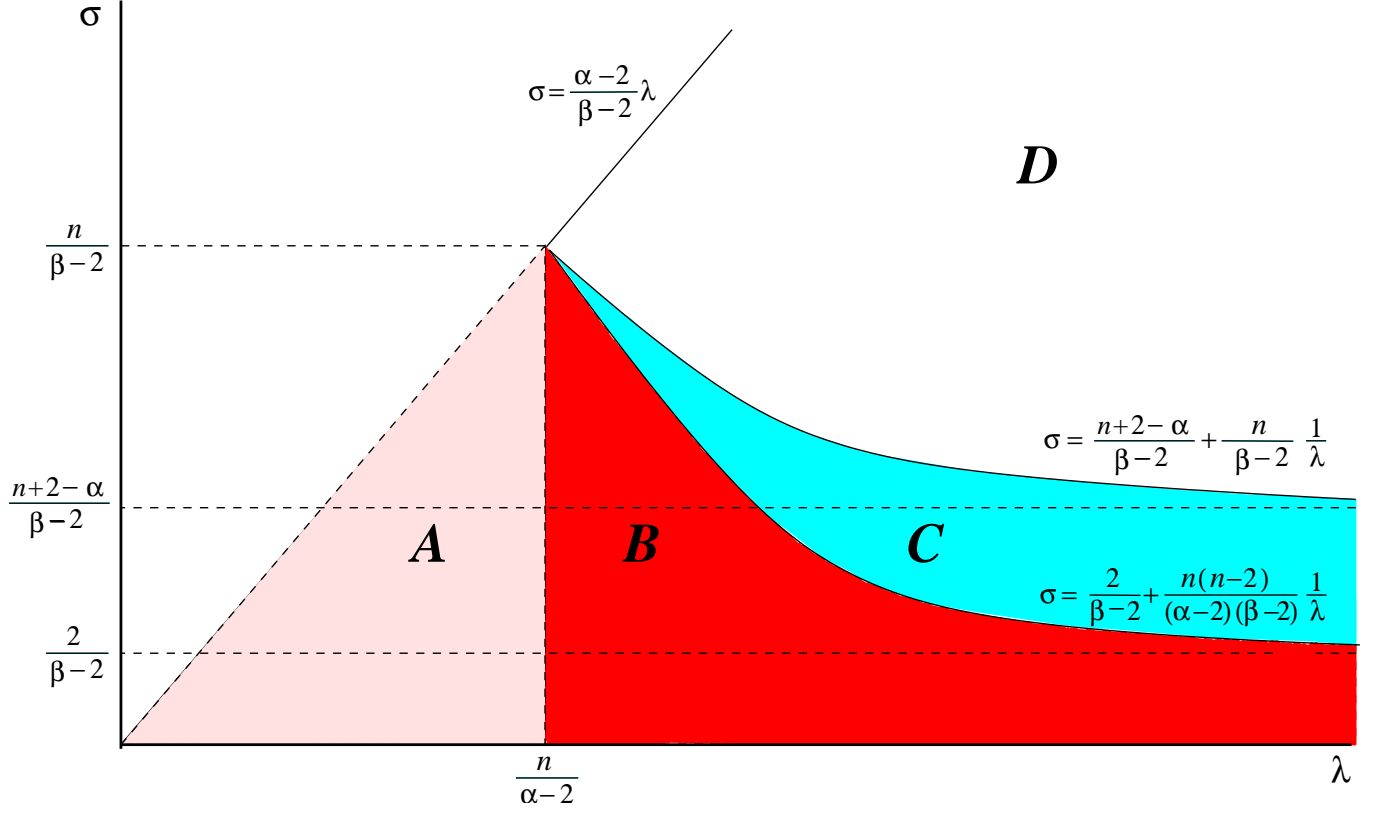


Figure 1: Graph of regions A , B , C and D .

Note that A , B , C , and D are two dimensional regions in the $\lambda\sigma$ -plane whereas E is the curve separating C and D . (See Figure 1.)

The following theorem deals with the case that $(\lambda, \sigma) \in A$.

Theorem 2.3. Suppose $\alpha, \beta \in (2, n)$,

$$0 \leq \lambda \leq \frac{n}{\alpha-2} \quad \text{and} \quad 0 \leq \sigma < \frac{n}{\beta-2}. \quad (2.7)$$

Let u and v be $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions of (1.1). Then u and v are both harmonically bounded, that is u and v satisfy (2.5).

By Remark 1 the bounds (2.5) for u and v in Theorem 2.3 are optimal.

The following theorem deals with the case that $(\lambda, \sigma) \in B$.

Theorem 2.4. Suppose $\alpha, \beta \in (2, n)$,

$$\lambda > \frac{n}{\alpha-2} \quad \text{and} \quad 0 \leq \sigma \leq \frac{2}{\beta-2} + \frac{n(n-2)}{(\alpha-2)(\beta-2)} \frac{1}{\lambda}.$$

Let u and v be $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions of (1.1). Then

$$u(x) = o\left(|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0$$

and v is harmonically bounded, that is

$$v(x) = \mathcal{O}(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0.$$

The following theorem deals with the case that $(\lambda, \sigma) \in C$.

Theorem 2.5. *Suppose $\alpha, \beta \in (2, n)$,*

$$\lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad \frac{2}{\beta - 2} + \frac{n(n-2)}{(\alpha-2)(\beta-2)} \frac{1}{\lambda} < \sigma < \frac{n+2-\alpha}{\beta-2} + \frac{n}{\beta-2} \frac{1}{\lambda}.$$

Let u and v be $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions of (1.1). Then

$$u(x) = o\left(|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0$$

and

$$v(x) = o\left(|x|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-2]}{n}}\right) \quad \text{as } x \rightarrow 0.$$

By the following theorem the bounds for u and v in Theorems 2.4 and 2.5 are optimal.

Theorem 2.6. *Suppose $\alpha, \beta \in (2, n)$,*

$$\lambda > \frac{n}{\alpha - 2}, \quad \text{and} \quad 0 < \sigma < \frac{n}{\beta - 2}.$$

Let $h : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\lim_{t \rightarrow 0^+} h(t) = 0$. Then there exist $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions u and v of (1.1) such that

$$u(x) \neq \mathcal{O}\left(h(|x|)|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0$$

and

$$v(x) \neq \mathcal{O}\left(h(|x|)\left[|x|^{-(n-2)} + |x|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-2]}{n}}\right]\right) \quad \text{as } x \rightarrow 0.$$

The following theorem deals with the case that $(\lambda, \sigma) \in D$. In this case there exist pointwise bounds for neither u nor v .

Theorem 2.7. *Suppose $\alpha, \beta \in (2, n)$,*

$$\lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad \sigma > \frac{n+2-\alpha}{\beta-2} + \frac{n}{\beta-2} \frac{1}{\lambda}. \quad (2.8)$$

Let $h : (0, 1) \rightarrow (0, \infty)$ be a continuous function such that $\lim_{t \rightarrow 0^+} h(t) = \infty$. Then there exist $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions u and v of (1.1) such that

$$u(x) \neq \mathcal{O}(h(|x|)) \quad \text{as } x \rightarrow 0 \quad (2.9)$$

and

$$v(x) \neq \mathcal{O}(h(|x|)) \quad \text{as } x \rightarrow 0. \quad (2.10)$$

From Theorems 2.1, 2.3 and 2.7 we find:

Corollary 2.8. *Let $\alpha \in (0, n)$ and $\lambda \geq 0$. Consider the inequality*

$$0 \leq -\Delta u \leq \left(\frac{1}{|x|^\alpha} * u\right)^\lambda \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n, n \geq 3. \quad (2.11)$$

(i) If $\lambda(\alpha - 2) < n$ then any $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solution u of (2.11) satisfies

$$u(x) = \mathcal{O}(|x|^{-(n-2)}) \quad \text{as } x \rightarrow 0.$$

(ii) If $\lambda(\alpha - 2) > n$ then (2.11) admits $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions which are arbitrarily large around the origin in the following sense: for any continuous function $h : (0, 1) \rightarrow (0, \infty)$ satisfying $\lim_{t \rightarrow 0^+} h(t) = \infty$, there exists a $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solution u of (2.11) such that

$$u(x) \neq \mathcal{O}(h(|x|)) \quad \text{as } x \rightarrow 0.$$

A first tool we use in our approach to (1.1) is an integral representation formula for nonnegative superharmonic functions in punctured balls due to Brezis and Lions [2] (see also [3, 7, 22, 23] where representation formulae for various kinds of differential operators are deduced). Another important tool in our approach is Proposition 3.1 which provides pointwise estimates for nonlinear potentials of Havin-Maz'ya type. Further, various integral estimates will be employed as stated in Section 3.2. The optimality of the pointwise bounds obtained in our main results will be achieved by constructing solutions u and v of (1.1) satisfying suitable coupled conditions on a countable sequence of balls that concentrate at the origin. At this stage we leave open the question of (non)existence of pointwise bounds for (λ, σ) on the curve E defined above.

The remainder of the paper is organized as follows: In Subsection 2.2 we state our main results for the system (1.6). In Section 3 we collect various pointwise and integral estimates for some quantities which will frequently appear in the course of our proofs. Sections 4–10 contain the proofs of our main results. Theorem 6.1, which deals with the system (1.6), is a crucial result, from which the optimal bounds for positive solutions of the systems (1.1) and (1.6) easily follow.

2.2 Results for system (1.6)

We now state our results for the system (1.6) when $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (2, n + 2)$. As in Subsection 2.1, we can assume that (2.6) holds. Let the regions A – D be defined as in Subsection 2.1.

The following theorem deals with the case $(\lambda, \sigma) \in A$.

Theorem 2.9. *Suppose $\alpha, \beta \in (2, n + 2)$,*

$$0 \leq \lambda \leq \frac{n}{\alpha - 2} \quad \text{and} \quad 0 \leq \sigma < \frac{n}{\beta - 2}.$$

Let f and g be $L^1(B_1(0))$ solutions of (1.6) where M is a positive constant. Then

$$f(x) = \mathcal{O}\left(|x|^{-\lambda(\alpha-2)}\right) \quad \text{as } x \rightarrow 0$$

and

$$g(x) = \mathcal{O}\left(|x|^{-\sigma(\beta-2)}\right) \quad \text{as } x \rightarrow 0.$$

The following theorem deals with the case $(\lambda, \sigma) \in B \cup C$.

Theorem 2.10. *Suppose $\alpha, \beta \in (2, n + 2)$,*

$$\lambda > \frac{n}{\alpha - 2} \quad \text{and} \quad 0 \leq \sigma < \frac{n + 2 - \alpha}{\beta - 2} + \frac{n}{\beta - 2} \frac{1}{\lambda}.$$

Let f and g be $L^1(B_1(0))$ solutions of (1.6) where M is a positive constant. Then

$$f(x) = \mathcal{O}\left(|x|^{-\lambda(\alpha-2)}\right) \quad \text{as } x \rightarrow 0$$

and

$$g(x) = o\left(|x|^{-\frac{\lambda(\alpha-2)\sigma(\beta-2)}{n}}\right) \quad \text{as } x \rightarrow 0.$$

By the following result the estimates for f and g in Theorems 2.9 and 2.10 are optimal.

Theorem 2.11. Suppose $\varepsilon > 0$, $\alpha, \beta \in (2, n+2)$,

$$\lambda \geq 0 \quad \text{and} \quad 0 < \sigma < \frac{n}{\beta-2}.$$

Let $h : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\lim_{t \rightarrow 0^+} h(t) = 0$. Then there exist solutions

$$f, g \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n) \quad (2.12)$$

of the system

$$\begin{cases} 0 \leq f(x) \leq \varepsilon \left(|x|^{-(\alpha-2)} + \int_{|y|<\varepsilon} \frac{g(y) dy}{|x-y|^{\alpha-2}} \right)^\lambda \\ 0 \leq g(x) \leq \varepsilon \left(|x|^{-(\beta-2)} + \int_{|y|<\varepsilon} \frac{f(y) dy}{|x-y|^{\beta-2}} \right)^\sigma \end{cases} \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}, n \geq 3, \quad (2.13)$$

such that

$$\limsup_{x \rightarrow 0} |x|^{\lambda(\alpha-2)} f(x) > 0 \quad (2.14)$$

and

$$g(x) \neq \mathcal{O}\left(h(x) \left[|x|^{-\sigma(\beta-2)} + |x|^{-\frac{\lambda(\alpha-2)\sigma(\beta-2)}{n}} \right] \right) \quad \text{as } x \rightarrow 0. \quad (2.15)$$

The following theorem deals with the case that $(\lambda, \sigma) \in D$. In this case there exist pointwise bounds for neither f nor g .

Theorem 2.12. Suppose $\varepsilon > 0$, $\alpha, \beta \in (2, n+2)$,

$$\lambda > \frac{n}{\alpha-2} \quad \text{and} \quad \sigma > \frac{n+2-\alpha}{\beta-2} + \frac{n}{\beta-2} \frac{1}{\lambda}. \quad (2.16)$$

Let $h : (0, 1) \rightarrow (0, \infty)$ be a continuous function such that $\lim_{t \rightarrow 0^+} h(t) = \infty$. Then there exist solutions

$$f, g \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n) \quad (2.17)$$

of the system (2.13) such that

$$f(x) \neq \mathcal{O}(h(|x|)) \quad \text{as } x \rightarrow 0 \quad (2.18)$$

and

$$g(x) \neq \mathcal{O}(h(|x|)) \quad \text{as } x \rightarrow 0. \quad (2.19)$$

3 Preliminary results

3.1 Nonlinear Riesz potentials

Let B be a ball in \mathbb{R}^n , $n \geq 3$, and $f \in L^\infty(B)$ be a nonnegative function. For any $a \in (0, n)$ we define the Riesz potential $\mathbf{I}_a f$ of order a by

$$\mathbf{I}_a f(x) = \int_B \frac{f(y)}{|x-y|^{n-a}} dy \quad \text{for all } x \in B.$$

We also set

$$\mathbf{U}_{a,b,\sigma} f := \mathbf{I}_a ((\mathbf{I}_b f)^\sigma),$$

where $a, b \in (0, n)$. If $a = b$ then $\mathbf{U}_{a,a,\sigma} f$ is the Havin-Maz'ya potential [10].

Proposition 3.1. *Let $a, b \in (0, n)$ and $\sigma > \frac{a}{n-b}$. Then there exists a constant $C = C(n, \sigma, a, b) > 0$ such that*

$$\|\mathbf{U}_{a,b,\sigma} f\|_\infty \leq C \|f\|_1^{\frac{a+b\sigma}{n}} \|f\|_\infty^{\frac{\sigma(n-b)-a}{n}} \quad \text{for all } f \in L^\infty(B), f \geq 0.$$

Proof. Let us first recall Hedberg's inequality [11]

$$\|\mathbf{I}_\gamma f\|_\infty \leq C(n, \gamma, p) \|f\|_p^{\frac{\gamma p}{n}} \|f\|_\infty^{1-\frac{\gamma p}{n}} \quad \text{for all } 0 < \gamma < n, 1 \leq p < \frac{n}{\gamma}. \quad (3.1)$$

Let $g = (\mathbf{I}_b f)^\sigma$. Using (3.1) we have

$$\|\mathbf{I}_a g\|_\infty \leq C(n, a, p) \|g\|_p^{\frac{ap}{n}} \|g\|_\infty^{1-\frac{ap}{n}} \quad \text{for all } 1 \leq p < \frac{n}{a} \quad (3.2)$$

and

$$\|\mathbf{I}_b f\|_\infty \leq C(n, b) \|f\|_1^{\frac{b}{n}} \|f\|_\infty^{1-\frac{b}{n}}.$$

This last estimate implies

$$\|g\|_\infty = \|\mathbf{I}_b f\|_\infty^\sigma \leq C \|f\|_1^{\frac{\sigma b}{n}} \|f\|_\infty^{\sigma(1-\frac{b}{n})}. \quad (3.3)$$

Since $\sigma > \frac{a}{n-b}$, we can find $s \in (1, n/b)$ and $p \in (1, n/a)$ such that

$$p\sigma = \frac{ns}{n-bs}. \quad (3.4)$$

By standard Riesz potential estimates (see [9, Lemma 7.12]) we have

$$\|g\|_p = \|\mathbf{I}_b f\|_{p\sigma}^\sigma \leq C \|f\|_s^\sigma. \quad (3.5)$$

We now use (3.3) and (3.5) in (3.2) to deduce

$$\|\mathbf{U}_{a,b,\sigma} f\|_\infty = \|\mathbf{I}_a g\|_\infty \leq C \|f\|_s^{\frac{ap\sigma}{n}} \|f\|_1^{\frac{\sigma b}{n}(1-\frac{ap}{n})} \|f\|_\infty^{\sigma(1-\frac{b}{n})(1-\frac{ap}{n})}. \quad (3.6)$$

Finally, using the estimate

$$\|f\|_s \leq \|f\|_1^{\frac{1}{s}} \|f\|_\infty^{\frac{s-1}{s}}$$

in (3.6) we obtain

$$\begin{aligned} \|\mathbf{U}_{a,b,\sigma} f\|_\infty &\leq C \|f\|_1^{\frac{ap\sigma}{ns} + \frac{\sigma b}{n}(1-\frac{ap}{n})} \|f\|_\infty^{\frac{ap\sigma}{n} \frac{s-1}{s} + \sigma(1-\frac{b}{n})(1-\frac{ap}{n})} \\ &= C \|f\|_1^{\frac{a+b\sigma}{n}} \|f\|_\infty^{\frac{\sigma(n-b)-a}{n}} \end{aligned}$$

by (3.4). □

3.2 Further estimates

In this part we collect some results which will be used in our proofs. A very important tool in our approach is the following result due to Brezis and Lions which presents a representation formula for nonnegative superharmonic functions in a punctured ball of \mathbb{R}^n .

Lemma 3.2. (see [2]) *Let u be a C^2 nonnegative superharmonic function in $B_{2r}(0) \setminus \{0\} \subset \mathbb{R}^n$, $n \geq 3$, for some $r > 0$. Then*

$$u, -\Delta u \in L^1(B_r(0))$$

and there exist $m \geq 0$, $c = c(n) > 0$ and a bounded harmonic function $h : B_r(0) \rightarrow \mathbb{R}$ such that

$$u(x) = m|x|^{2-n} + c \int_{|y|<r} \frac{-\Delta u(y)}{|x-y|^{n-2}} dy + h(x) \quad \text{for } 0 < |x| < r.$$

Lemma 3.3. (see [8, Lemma 5.1]) *Let $\varphi : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\lim_{t \rightarrow 0^+} \varphi(t) = 0$. Let $\{x_j\} \subset \mathbb{R}^n$, $n \geq 3$, be a sequence satisfying*

$$0 < 4|x_{j+1}| < |x_j| < \frac{1}{2} \quad \text{and} \quad \sum_{j=1}^{\infty} \varphi(|x_j|) < \infty, \quad (3.7)$$

and $\{r_j\} \subset \mathbb{R}$ be such that $0 < r_j \leq |x_j|/2$.

Then there exist a positive constant $A = A(n)$ and a positive function $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$0 \leq -\Delta u \leq \frac{\varphi(|x_j|)}{r_j^n} \quad \text{in } B_{r_j}(x_j), \quad (3.8)$$

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^n \setminus \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \right), \quad (3.9)$$

$$u \geq 1 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad (3.10)$$

and

$$u \geq \frac{A\varphi(|x_j|)}{r_j^{n-2}} \quad \text{in } B_{r_j}(x_j). \quad (3.11)$$

Lemma 3.4. *Let $\alpha \in (0, n)$ and $v \in L^1(B_1(0))$ be a nonnegative function such that*

$$v(x) = \mathcal{O}(|x|^{-\gamma}) \quad \text{as } x \rightarrow 0$$

for some $\gamma \geq 0$. Then

$$\int_{|y|<1} \frac{v(y) dy}{|x-y|^\alpha} = \mathcal{O}(|x|^{-\alpha}) + o\left(|x|^{-\frac{\gamma\alpha}{n}}\right) \quad \text{as } x \rightarrow 0. \quad (3.12)$$

Proof. Choose $C > 0$ and $R \in (0, 1/4)$ such that

$$v(y) \leq C|y|^{-\gamma} \quad \text{for } 0 < |y| < 2R.$$

Let $\{x_j\} \subset B_R(0) \setminus \{0\}$ be a sequence which converges to 0. Then

$$v(y) \leq C|x_j|^{-\gamma} \quad \text{for } |y - x_j| < \frac{|x_j|}{2} \quad (3.13)$$

and it suffices to prove the estimate (3.12) with x replaced with x_j .

Define $r_j \in (0, |x_j|/2)$ by

$$\int_{|y-x_j|<r_j} C|x_j|^{-\gamma} dy = \int_{|y-x_j|<\frac{|x_j|}{2}} v(y) dy \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (3.14)$$

Then $r_j = o(|x_j|^{\gamma/n})$ as $j \rightarrow \infty$. Also, using (3.13) and (3.14) we have

$$\begin{aligned} \int_{|y-x_j|<r_j} \frac{C|x_j|^{-\gamma} - v(y)}{|y-x_j|^\alpha} dy &\geq \int_{|y-x_j|<r_j} \frac{C|x_j|^{-\gamma} - v(y)}{r_j^\alpha} dy \\ &= \int_{r_j < |y-x_j| < \frac{|x_j|}{2}} \frac{v(y)}{r_j^\alpha} dy \\ &\geq \int_{r_j < |y-x_j| < \frac{|x_j|}{2}} \frac{v(y)}{|y-x_j|^\alpha} dy \end{aligned}$$

which yields

$$\int_{|y-x_j|<\frac{|x_j|}{2}} \frac{v(y) dy}{|y-x_j|^\alpha} \leq \int_{|y-x_j|<r_j} \frac{C|x_j|^{-\gamma} dy}{|y-x_j|^\alpha}.$$

Using this last estimate, for j large we have

$$\begin{aligned} \int_{|y|<1} \frac{v(y) dy}{|x_j - y|^\alpha} &= \int_{|y|<1, |y-x_j|>\frac{|x_j|}{2}} \frac{v(y) dy}{|y-x_j|^\alpha} + \int_{|y-x_j|<\frac{|x_j|}{2}} \frac{v(y) dy}{|y-x_j|^\alpha} \\ &\leq C|x_j|^{-\alpha} + \int_{|y-x_j|<\frac{|x_j|}{2}} \frac{v(y) dy}{|y-x_j|^\alpha} \\ &\leq C \left[|x_j|^{-\alpha} + \int_{|y-x_j|<r_j} \frac{|x_j|^{-\gamma} dy}{|y-x_j|^\alpha} \right] \\ &\leq C \left[|x_j|^{-\alpha} + r_j^{n-\alpha} |x_j|^{-\gamma} \right] \\ &= \mathcal{O}(|x_j|^{-\alpha}) + o\left(|x_j|^{-\frac{\gamma\alpha}{n}}\right) \quad \text{as } j \rightarrow \infty. \end{aligned}$$

□

Corollary 3.5. *Let u be a C^2 nonnegative function in $B_r(0) \setminus \{0\} \subset \mathbb{R}^n$, $n \geq 3$, $r > 0$ such that for some $\gamma \geq 0$ we have*

$$0 \leq -\Delta u \leq C|x|^{-\gamma} \quad \text{for } 0 < |x| < r.$$

Then

$$u(x) = \mathcal{O}(|x|^{2-n}) + o\left(|x|^{-\frac{\gamma(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0.$$

Proof. We apply the representation formula in Lemma 3.2 and then Lemma 3.4 with $v = -\Delta u$ and $\alpha = n - 2$. □

Lemma 3.6. *Let $\alpha, \beta < n$. Then there exists a constant $C = C(n, \alpha, \beta) > 0$ such that*

$$\int_{|y|<2} \frac{dy}{|x-y|^\beta |y|^\alpha} \leq \begin{cases} \frac{C}{|x|^{\alpha+\beta-n}} & \text{if } \alpha + \beta > n, \\ C \ln \frac{2}{|x|} & \text{if } \alpha + \beta = n, \\ C & \text{if } \alpha + \beta < n. \end{cases} \quad \text{for } 0 < |x| < 1.$$

Proof. We could use the convolution formula (see Stein [20, pg. 118])

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^\beta |y|^\alpha} = \frac{C(n, \alpha, \beta)}{|x|^{\alpha+\beta-n}} \quad \text{for all } x \in \mathbb{R}^n, \quad (3.15)$$

which holds whenever $\alpha + \beta > n$. However, we shall give here a direct and simpler proof.

Let $x \in B_1(0) \setminus \{0\}$ and $r = |x|$. Under the change of variable $x = r\xi$, $y = r\eta$, we have $|\xi| = 1$ and

$$\begin{aligned} \int_{|y|<2} \frac{dy}{|x-y|^\beta |y|^\alpha} &= r^{n-\alpha-\beta} \int_{|\eta|<\frac{2}{r}} \frac{d\eta}{|\xi-\eta|^\beta |\eta|^\alpha} \\ &= r^{n-\alpha-\beta} \left[\int_{|\eta|<2} \frac{d\eta}{|\xi-\eta|^\beta |\eta|^\alpha} + \int_{2<|\eta|<\frac{2}{r}} \frac{d\eta}{|\xi-\eta|^\beta |\eta|^\alpha} \right] \\ &= r^{n-\alpha-\beta} \left[C(n, \alpha, \beta) + \int_{2<|\eta|<\frac{2}{r}} \frac{d\eta}{|\xi-\eta|^\beta |\eta|^\alpha} \right] \\ &\leq Cr^{n-\alpha-\beta} \left[1 + \int_{2<|\eta|<\frac{2}{r}} \frac{d\eta}{|\eta|^{\alpha+\beta}} \right] \\ &\leq \begin{cases} \frac{C}{r^{\alpha+\beta-n}} & \text{if } \alpha + \beta > n, \\ C \ln \frac{2}{r} & \text{if } \alpha + \beta = n, \\ C & \text{if } \alpha + \beta < n. \end{cases} \end{aligned}$$

□

Corollary 3.7. *Let $\alpha, \beta < n$ and $R > 0$. Then there exists a constant $C = C(n, \alpha, \beta) > 0$ such that for all $x, z \in B_R(0)$, $x \neq z$ we have*

$$\int_{|y|<R} \frac{dy}{|x-y|^\beta |y-z|^\alpha} \leq \begin{cases} \frac{C}{|x-z|^{\alpha+\beta-n}} & \text{if } \alpha + \beta > n, \\ C \ln \frac{4R}{|x-z|} & \text{if } \alpha + \beta = n, \\ \frac{C}{R^{\alpha+\beta-n}} & \text{if } \alpha + \beta < n. \end{cases} \quad (3.16)$$

Proof. Under the change of variables $\xi = \frac{x-z}{2R}$, $\eta = \frac{y-z}{2R}$ we find $\xi \in B_1(0) \setminus \{0\}$ and thus by Lemma 3.6 we have

$$\begin{aligned} \int_{|y|<R} \frac{dy}{|x-y|^\beta |y-z|^\alpha} &\leq (2R)^{n-\alpha-\beta} \int_{|\eta|<2} \frac{d\eta}{|\xi-\eta|^\beta |\eta|^\alpha} \\ &\leq \begin{cases} \frac{C}{(2R|\xi|)^{\alpha+\beta-n}} & \text{if } \alpha + \beta > n, \\ C \ln \frac{2}{|\xi|} & \text{if } \alpha + \beta = n, \\ \frac{C}{(2R)^{\alpha+\beta-n}} & \text{if } \alpha + \beta < n. \end{cases} \end{aligned}$$

This clearly implies (3.16). \square

Lemma 3.8. *Let $\sigma > 0$ and $\gamma \in (0, n)$. There exists a constant $C = C(n, \sigma, \gamma) > 0$ such that*

$$\int_{|y|<1} \frac{\ln^\sigma \frac{4}{|y-z|}}{|x-y|^\gamma} dy \leq C \quad \text{for all } x, z \in B_1(0).$$

Proof. This follows from Riesz potential estimates (see [9, Lemma 7.12]). \square

Lemma 3.9. *Suppose u and v are $C^2(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ positive solutions of (1.1) where $\lambda, \sigma \geq 0$ and $\alpha, \beta \in (0, n)$. Then*

$$-\Delta u, -\Delta v \in L^1(B_1(0)) \quad (3.17)$$

and for some positive constant C we have

$$\begin{cases} 0 \leq -\Delta u(x) \leq C \left(\int_{|y|<1} \frac{v(y) dy}{|x-y|^\alpha} \right)^\lambda \\ 0 \leq -\Delta v(x) \leq C \left(\int_{|y|<1} \frac{u(y) dy}{|x-y|^\beta} \right)^\sigma \end{cases} \quad \text{for } 0 < |x| < 1, \quad (3.18)$$

$$\begin{cases} u(x) \leq C \left(|x|^{2-n} + \int_{|y|<1} \frac{-\Delta u(y) dy}{|x-y|^{n-2}} \right) \\ v(x) \leq C \left(|x|^{2-n} + \int_{|y|<1} \frac{-\Delta v(y) dy}{|x-y|^{n-2}} \right) \end{cases} \quad \text{for } 0 < |x| < 1, \quad (3.19)$$

and

$$\begin{cases} -\Delta u(x) \leq C \left[\int_{|y|<1} \frac{dy}{|y-x|^\alpha |y|^{n-2}} + \int_{|z|<1} -\Delta v(z) \left(\int_{|y|<1} \frac{dy}{|x-y|^\alpha |y-z|^{n-2}} \right) dz \right]^\lambda \\ -\Delta v(x) \leq C \left[\int_{|y|<1} \frac{dy}{|y-x|^\beta |y|^{n-2}} + \int_{|z|<1} -\Delta u(z) \left(\int_{|y|<1} \frac{dy}{|x-y|^\beta |y-z|^{n-2}} \right) dz \right]^\sigma \end{cases} \quad (3.20)$$

for $0 < |x| < 1$.

Proof. Lemma 3.2 implies (3.17) holds. Since

$$\int_{|y|<1} \frac{v(y) dy}{|x-y|^\alpha} > \int_{|y|<1} \frac{v(y) dy}{2^\alpha} =: C_1 > 0 \quad \text{for } |x| < 1$$

and

$$\int_{|y|>1} \frac{v(y) dy}{|x-y|^\alpha} \leq \int_{1<|y|<2} \frac{\max_{1\leq|y|\leq 2} v(y)}{|x-y|^\alpha} dy + \int_{|y|>2} v(y) dy \leq C_2 \quad \text{for } |x| < 1$$

we see that

$$\int_{\mathbb{R}^n} \frac{v(y) dy}{|x-y|^\alpha} \leq \left(1 + \frac{C_2}{C_1}\right) \int_{|y|<1} \frac{v(y) dy}{|x-y|^\alpha} \quad \text{for } |x| < 1.$$

Thus the first line of (3.18) follows from (1.1). The second line of (3.18) is proved similarly.

Inequalities (3.19) follow from Lemma 3.2. Substituting (3.19) in (3.18) we get (3.20). \square

4 Proof of Theorem 2.1

By Lemma 3.9, u and v satisfy (3.17)–(3.20).

If $\beta < 2$ then (3.20), (3.17), Lemma 3.6 and Corollary 3.7 yield

$$-\Delta v(x) \leq C \left[1 + \int_{|z|<1} -\Delta u(z) dz \right]^\sigma \leq C \quad \text{for } 0 < |x| < 1$$

and from Corollary 3.5 we find $v(x)$ satisfies (2.2).

Assume next that $\beta = 2$. Then using Lemma 3.6 and Corollary 3.7 we obtain from (3.20) that

$$-\Delta v(x) \leq C \left[\ln \frac{4}{|x|} + \int_{|z|<1} \left(\ln \frac{4}{|x-z|} \right) (-\Delta u(z)) dz \right]^\sigma \quad \text{for } 0 < |x| < 1. \quad (4.1)$$

Since increasing σ increases the right side of (4.1), it follows from (4.1) that there exists $\gamma > 1$ such that u and v satisfy

$$-\Delta v(x) \leq C \left[\ln \frac{4}{|x|} + \int_{|z|<1} \left(\ln \frac{4}{|x-z|} \right) (-\Delta u(z)) dz \right]^\gamma \quad \text{for } 0 < |x| < 1.$$

Thus by Jensen's inequality, we have

$$\begin{aligned}
-\Delta v(x) &\leq C \left[\ln^\gamma \frac{2}{|x|} + \left(\int_{|z|<1} \left(\ln \frac{4}{|x-z|} \right) (-\Delta u(z)) dz \right)^\gamma \right] \\
&= C \left[\ln^\gamma \frac{2}{|x|} + \|\Delta u\|_{L^1(B_1(0))}^\gamma \left(\int_{|z|<1} \left(\ln \frac{4}{|x-z|} \right) \frac{-\Delta u(z)}{\|\Delta u\|_{L^1(B_1(0))}} dz \right)^\gamma \right] \\
&\leq C \left[\ln^\gamma \frac{2}{|x|} + \int_{|z|<1} \left(\ln^\gamma \frac{4}{|x-z|} \right) (-\Delta u(z)) dz \right] \quad \text{for } 0 < |x| < 1.
\end{aligned}$$

This last estimate combined with (3.19), (3.17), and Lemma 3.8 yields

$$\begin{aligned}
v(x) &\leq C \left[|x|^{2-n} + \int_{|y|<1} \frac{-\Delta v(y)}{|x-y|^{n-2}} dy \right] \\
&\leq C \left[|x|^{2-n} + \int_{|y|<1} \frac{\ln^\gamma \frac{2}{|y|}}{|x-y|^{n-2}} dy + \int_{|z|<1} -\Delta u(z) \left(\int_{|y|<1} \frac{\ln^\gamma \frac{4}{|y-z|}}{|x-y|^{n-2}} dy \right) dz \right] \\
&\leq C \left[|x|^{2-n} + \|\Delta u\|_{L^1(B_1(0))} \right] \\
&\leq C |x|^{2-n} \quad \text{for } 0 < |x| < 1.
\end{aligned}$$

We have thus established (2.2) for $\beta \leq 2$. Now, from the first equation of (3.18) and Lemma 3.6 we find

$$\begin{aligned}
-\Delta u(x) &\leq C \left(\int_{|y|<1} \frac{v(y) dy}{|x-y|^\alpha} \right)^\lambda \leq C \left(\int_{|y|<1} \frac{dy}{|x-y|^\alpha |y|^{n-2}} \right)^\lambda \\
&\leq \begin{cases} C|x|^{-\lambda(\alpha-2)} & \text{if } \alpha > 2, \\ C \ln^\lambda \frac{2}{|x|} & \text{if } \alpha = 2, \\ C & \text{if } \alpha < 2. \end{cases} \quad \text{for } 0 < |x| < 1.
\end{aligned}$$

If $\alpha \leq 2$ we use (3.19) and Lemma 3.8 to deduce $u(x) = \mathcal{O}(|x|^{2-n})$ as $x \rightarrow 0$. If $\alpha > 2$ then we apply directly Corollary 3.5 to derive

$$u(x) = \mathcal{O}(|x|^{2-n}) + o\left(|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0,$$

and complete the proof of (2.1).

5 Proof of Theorem 2.2

Define $\varphi : (0, 1) \rightarrow (0, 1)$ by $\varphi = \sqrt{h}$. Let $\{x_j\} \subset \mathbb{R}^n$ be a sequence satisfying (3.7) and

$$r_j = |x_j|^{\frac{\lambda(\alpha-2)}{n}} \ll |x_j| \quad \text{as } j \rightarrow \infty. \quad (5.1)$$

By Lemma 3.3 there exist a positive constant $A = A(n)$ and a positive function $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ that satisfies (3.8)–(3.11). In particular, $-\Delta u \geq 0$ in $\mathbb{R}^n \setminus \{0\}$. Let

$$\hat{u} = u\chi + w(1 - \chi) \quad \text{and} \quad v = |x|^{-(n-2)}\chi + w(1 - \chi)$$

where $w \in C^\infty(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ is a positive function and $\chi \in C^\infty(\mathbb{R}^n)$ is a nonnegative function satisfying $\chi = 1$ in $B_2(0)$ and $\chi = 0$ in $\mathbb{R}^n \setminus B_3(0)$. Then $\hat{u}, v \in C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ by Lemma 3.2. Also, $\hat{u} = u$ and $v = |x|^{-(n-2)}$ in $B_2(0) \setminus \{0\}$. For simplicity of notation, we again denote \hat{u} by u . Since v is harmonic in $B_2(0) \setminus \{0\}$ we only need to check that u and v satisfy

$$0 \leq -\Delta u(x) \leq \left(\int_{\mathbb{R}^n} \frac{v(y) dy}{|x - y|^\alpha} \right)^\lambda \quad \text{for } 0 < |x| < 2 \quad (5.2)$$

and that (2.3) holds. In fact, owing to (3.9), we only need to check that (5.2) is valid in $\bigcup_{j=1}^\infty B_{r_j}(x_j)$.

For $x \in B_{r_j}(x_j)$ we have $x \in B_1(0)$ and

$$\int_{\mathbb{R}^n} \frac{v(y) dy}{|x - y|^\alpha} \geq \int_{|y-x| < \frac{|x|}{2}} \frac{dy}{|x - y|^\alpha |y|^{n-2}} \geq \frac{C}{|x|^{n-2}} \int_{|y-x| < \frac{|x|}{2}} \frac{dy}{|x - y|^\alpha} = \frac{C}{|x|^{\alpha-2}} > \frac{C}{|x_j|^{\alpha-2}}. \quad (5.3)$$

We now combine (3.8), (5.1) and (5.3) to obtain

$$-\Delta u(x) \leq \frac{\varphi(|x_j|)}{r_j^n} \leq \frac{1}{r_j^n} = \frac{1}{|x_j|^{\lambda(\alpha-2)}} \leq C \left(\int_{\mathbb{R}^n} \frac{v(y) dy}{|x - y|^\alpha} \right)^\lambda$$

for all $x \in B_{r_j}(x_j)$. This establishes (5.2). To check (2.3) we use (3.11), (5.1) and obtain

$$\begin{aligned} \frac{u(x_j)}{h(|x_j|)|x_j|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}} &\geq \frac{A\varphi(|x_j|)}{h(|x_j|)r_j^{n-2}|x_j|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}} \\ &= \frac{A}{\sqrt{h(|x_j|)}} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \end{aligned}$$

6 Proof of Theorems 2.3–2.5, 2.9, and 2.10

The theorems in the title of this section are either immediate consequences of the following theorem or follow very easily from it. Its proof is the crux of this paper. Specifically, estimates (6.2) and (6.3) immediately give Theorems 2.9 and 2.10, and, as we will see at the end of this section, Theorems 2.3–2.5 follow easily from estimates (6.4) and (6.5).

Theorem 6.1. *Assume $\alpha, \beta \in (2, n+2)$, $\lambda \geq 0$, and*

$$0 \leq \sigma < \min \left\{ \frac{n}{\beta-2}, \frac{n+2-\alpha}{\beta-2} + \frac{n}{\beta-2} \frac{1}{\lambda} \right\}. \quad (6.1)$$

Let f and g be $L^1(B_1(0))$ solutions of (1.6) where M is a positive constant. Then

$$f(x) = \mathcal{O}(|x|^{-\lambda(\alpha-2)}) \quad \text{as } x \rightarrow 0 \quad (6.2)$$

and

$$g(x) = \mathcal{O}\left(|x|^{-\sigma(\beta-2)}\right) + o\left(|x|^{-\frac{\lambda(\alpha-2)\sigma(\beta-2)}{n}}\right) \quad \text{as } x \rightarrow 0. \quad (6.3)$$

Also, for $2 < s < n + 2$ we have

$$\int_{|y|<1} \frac{f(y) dy}{|x-y|^{s-2}} = \mathcal{O}\left(|x|^{-(s-2)}\right) + o\left(|x|^{-\frac{\lambda(\alpha-2)(s-2)}{n}}\right) \quad \text{as } x \rightarrow 0 \quad (6.4)$$

and

$$\int_{|y|<1} \frac{g(y) dy}{|x-y|^{s-2}} = \mathcal{O}\left(|x|^{-(s-2)}\right) + o\left(|x|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-(n+2-s)]}{n}}\right) \quad \text{as } x \rightarrow 0. \quad (6.5)$$

In particular,

$$\int_{|y|<1} \frac{g(y) dy}{|x-y|^{\alpha-2}} = \mathcal{O}\left(|x|^{-(\alpha-2)}\right) \quad \text{as } x \rightarrow 0. \quad (6.6)$$

Proof. The estimate (6.6) follows from (6.5) and (6.1). Also, using Lemma 3.4, we see that (6.2) implies (6.4). Moreover, (6.4) with $s = \beta$ combined with (1.6) implies (6.3). Hence it remains only to prove (6.2) and (6.5).

We first prove (6.2). If $\lambda = 0$ then (6.2) follows immediately from (1.6). Hence we can assume for the proof of (6.2) that

$$\lambda > 0. \quad (6.7)$$

Moreover, since the estimate (6.2) for f does not depend on σ and since increasing σ weakens the conditions on f and g in the system (1.6), we can also assume for the proof of (6.2) that

$$\sigma > \frac{n+2-\alpha}{\beta-2}. \quad (6.8)$$

We divide the proof of (6.2) into two steps.

Step 1: For some $\gamma > n$ we have

$$f(x) = \mathcal{O}(|x|^{-\gamma}) \quad \text{as } x \rightarrow 0. \quad (6.9)$$

Let $\{x_j\} \subset \mathbb{R}^n$ be a sequence such that

$$0 < 4|x_{j+1}| < |x_j| < \frac{1}{2} \quad \text{for } j = 1, 2, \dots \quad (6.10)$$

To prove (6.9), it suffices to prove

$$f(x_j) = \mathcal{O}(|x_j|^{-\gamma}) \quad \text{as } j \rightarrow \infty. \quad (6.11)$$

Since

$$\int_{|y-x_j|>|x_j|/2, |y|<1} \frac{g(y) dy}{|x-y|^{\alpha-2}} \leq \left(\frac{4}{|x_j|}\right)^{\alpha-2} \int_{|y|<1} g(y) dy \leq C|x_j|^{2-\alpha} \quad \text{for } |x-x_j| < r_j := \frac{|x_j|}{4},$$

it follows from (1.6) that

$$f(x) \leq C \left[|x_j|^{2-\alpha} + \int_{|y-x_j| < \frac{|x_j|}{2}} \frac{g(y) dy}{|x-y|^{\alpha-2}} \right]^\lambda \quad \text{for } |x-x_j| < r_j; \quad (6.12)$$

and similarly

$$g(x) \leq C \left[|x_j|^{2-\beta} + \int_{|y-x_j| < \frac{|x_j|}{2}} \frac{f(y) dy}{|x-y|^{\beta-2}} \right]^\sigma \quad \text{for } |x-x_j| < r_j. \quad (6.13)$$

Let now $f_j, g_j : B_2(0) \rightarrow [0, \infty)$ be defined by

$$f_j(\xi) = r_j^n f(x_j + r_j \xi), \quad g_j(\xi) = r_j^n g(x_j + r_j \xi).$$

Since $f, g \in L^1(B_1(0))$ we have

$$\|f_j\|_{L^1(B_2(0))} \rightarrow 0, \quad \|g_j\|_{L^1(B_2(0))} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \quad (6.14)$$

Further, with the change of variable $y = x_j + r_j \zeta$ in (6.12) and (6.13) we find

$$r_j^{-n} f_j(\xi) = f(x_j + r_j \xi) \leq C |x_j|^{-\lambda(\alpha-2)} \left[1 + \int_{|\zeta| < 2} \frac{g_j(\zeta) d\zeta}{|\xi - \zeta|^{\alpha-2}} \right]^\lambda \quad \text{for } |\xi| < 1, \quad (6.15)$$

and

$$r_j^{-n} g_j(\zeta) = g(x_j + r_j \zeta) \leq C |x_j|^{-\sigma(\beta-2)} \left[1 + \int_{|\eta| < 2} \frac{f_j(\eta) d\eta}{|\zeta - \eta|^{\beta-2}} \right]^\sigma \quad \text{for } |\zeta| < 1. \quad (6.16)$$

For any $a \in (0, n)$, $r > 0$ and any $f \in L^1(B_r(0))$, $f \geq 0$ we denote by $\mathbf{I}_{a,r} f$ the Riesz potential

$$\mathbf{I}_{a,r} f(x) = \int_{B_r(0)} \frac{f(y) dy}{|x-y|^{n-a}}$$

and we define

$$\mathbf{U}_{a,b,\sigma;r} f := \mathbf{I}_{a,r} ((\mathbf{I}_{b,r} f)^\sigma).$$

Let $R \in (0, 1/2]$. By (6.14) we have

$$\int_{|\zeta| < 2} \frac{g_j(\zeta) d\zeta}{|\xi - \zeta|^{\alpha-2}} \leq C \left[\frac{1}{R^{\alpha-2}} + \int_{|\zeta| < 2R} \frac{g_j(\zeta) d\zeta}{|\xi - \zeta|^{\alpha-2}} \right] \quad \text{for } |\xi| < R.$$

In other words,

$$\int_{|\zeta| < 2} \frac{g_j(\zeta) d\zeta}{|\xi - \zeta|^{\alpha-2}} \leq C \left[\frac{1}{R^{\alpha-2}} + \mathbf{I}_{n+2-\alpha, 2R}(g_j)(\xi) \right] \quad \text{for } |\xi| < R. \quad (6.17)$$

Similarly, we find

$$\int_{|\eta|<2} \frac{f_j(\eta) d\eta}{|\zeta - \eta|^{\beta-2}} \leq C \left[\frac{1}{R^{\beta-2}} + \mathbf{I}_{n+2-\beta,4R}(f_j)(\zeta) \right] \quad \text{for } |\zeta| < 2R. \quad (6.18)$$

Combining (6.15), (6.16), (6.17) and (6.18) we deduce

$$f_j(\xi) \leq Cr_j^{n-\lambda(\alpha-2)} \left\{ \frac{1}{R^{\lambda(\alpha-2)}} + [\mathbf{I}_{n+2-\alpha,2R}(g_j)]^\lambda(\xi) \right\} \quad \text{for } |\xi| < R, \quad (6.19)$$

$$g_j(\zeta) \leq Cr_j^{n-\sigma(\beta-2)} \left\{ \frac{1}{R^{\sigma(\beta-2)}} + [\mathbf{I}_{n+2-\beta,4R}(f_j)]^\sigma(\zeta) \right\} \quad \text{for } |\zeta| < 2R. \quad (6.20)$$

Now, from (6.20) we find for all $\xi \in \mathbb{R}^n$ that

$$\begin{aligned} \mathbf{I}_{n+2-\alpha,2R}(g_j)(\xi) &\leq Cr_j^{n-\sigma(\beta-2)} \left\{ R^{n-\alpha+2-\sigma(\beta-2)} + \mathbf{I}_{n+2-\alpha,4R}[\mathbf{I}_{n+2-\beta,4R}(f_j)]^\sigma(\xi) \right\} \\ &= Cr_j^{n-\sigma(\beta-2)} \left\{ R^{n-\alpha+2-\sigma(\beta-2)} + \mathbf{U}_{n+2-\alpha,n+2-\beta,\sigma;4R}(f_j)(\xi) \right\}. \end{aligned}$$

It therefore follows from (6.19) that there exists a positive constant a which depends only on $n, \alpha, \beta, \lambda$, and σ such that

$$f_j(\xi) \leq \frac{C}{(Rr_j)^a} \left\{ 1 + [\mathbf{V}(f_j)(\xi)]^\lambda \right\} \quad \text{for } |\xi| < R \leq \frac{1}{2}, \quad (6.21)$$

where

$$\mathbf{V}(f) := \mathbf{U}_{n+2-\alpha,n+2-\beta,\sigma;4R}(f).$$

At this stage, to prove for some $\gamma > n$ that (6.11) holds, it suffices to show that for some $\gamma > 0$ the sequence $\{r_j^\gamma f_j(0)\}$ is bounded. This will be achieved by means of the following auxiliary result.

Lemma 6.2. *Suppose the sequence*

$$\{r_j^\gamma f_j\} \quad \text{is bounded in } L^p(B_{4R}(0)) \quad (6.22)$$

for some constants $\gamma \geq 0$, $p \in [1, \infty)$, and $R \in (0, 1/2]$. Let $\delta = \gamma\lambda\sigma + a$ where a is as in (6.21). Then either the sequence

$$\{r_j^\delta f_j\} \quad \text{is bounded in } L^\infty(B_R(0)) \quad (6.23)$$

or there exists a positive constant $C_0 = C_0(n, \lambda, \sigma, \alpha, \beta)$ such that the sequence

$$\{r_j^\delta f_j\} \quad \text{is bounded in } L^q(B_R(0))$$

for some $q \in (p, \infty)$ satisfying

$$\frac{1}{p} - \frac{1}{q} > C_0. \quad (6.24)$$

Proof of Lemma 6.2. It follows from (6.1) that there exists $\varepsilon = \varepsilon(n, \lambda, \sigma, \alpha, \beta) > 0$ such that

$$\alpha, \beta < n + 2 - \varepsilon \quad \text{and} \quad 0 \leq \sigma < \min \left\{ \frac{n}{\beta - 2 + \varepsilon}, \frac{n + \lambda(n + 2 - \alpha - \varepsilon)}{\lambda(\beta - 2 + \varepsilon)} \right\}. \quad (6.25)$$

By (6.21) we have

$$r_j^\delta f_j(\xi) \leq \frac{C}{R^a} \left(1 + ((\mathbf{V}(r_j^\gamma f_j))(\xi))^\lambda \right) \quad \text{for } |\xi| < R. \quad (6.26)$$

We can assume

$$p \leq n/(n+2-\beta) \quad (6.27)$$

for otherwise from Riesz potential estimates (see [9, Lemma 7.12]) and (6.22) we find that the sequence $\{I_{n+2-\beta,4R}(r_j^\gamma f_j)\}$ is bounded in $L^\infty(B_{4R}(0))$ and hence by (6.26) we see that (6.23) holds.

Define p_1 by

$$\frac{1}{p} - \frac{1}{p_1} = \frac{n+2-\beta-\varepsilon}{n}, \quad (6.28)$$

where ε is as in (6.25). By (6.27), $p_1 \in (p, \infty)$ and by Riesz potential estimates we have

$$\|(\mathbf{I}_{n+2-\beta,4R}f_j)^\sigma\|_{p_1/\sigma} = \|\mathbf{I}_{n+2-\beta,4R}f_j\|_{p_1}^\sigma \leq C\|f_j\|_p^\sigma \quad (6.29)$$

where $\|\cdot\|_p := \|\cdot\|_{L^p(B_{4R}(0))}$. Since, by (6.25),

$$\frac{1}{p_1} = \frac{1}{p} - \frac{n+2-\beta-\varepsilon}{n} \leq 1 - \frac{n+2-\beta-\varepsilon}{n} = \frac{\beta-2+\varepsilon}{n} < \frac{1}{\sigma},$$

we have

$$p_1/\sigma > 1. \quad (6.30)$$

We can assume

$$p_1/\sigma \leq n/(n+2-\alpha) \quad (6.31)$$

for otherwise by Riesz potential estimates and (6.29) we have

$$\begin{aligned} \|\mathbf{V}(r_j^\gamma f_j)\|_\infty &= \|\mathbf{U}_{n+2-\alpha,n+2-\beta,\sigma;4R}(r_j^\gamma f_j)\|_\infty \\ &\leq C\|(\mathbf{I}_{n+2-\beta,4R}(r_j^\gamma f_j))^\sigma\|_{\frac{p_1}{\sigma}} \\ &\leq C\|r_j^\gamma f_j\|_p^\sigma \end{aligned}$$

which is a bounded sequence by (6.22). Hence (6.26) implies (6.23).

Define p_2 by

$$\frac{\sigma}{p_1} - \frac{1}{p_2} = \frac{n+2-\alpha-\varepsilon}{n} \quad \text{and let} \quad q = \frac{p_2}{\lambda}. \quad (6.32)$$

By (6.30) and (6.31), $p_2 \in (1, \infty)$ and by Riesz potential estimates

$$\begin{aligned} \|\mathbf{V}(f_j)^\lambda\|_q &= \|\mathbf{U}_{n+2-\alpha,n+2-\beta,\sigma;4R}(f_j)\|_{p_2}^\lambda \\ &\leq C\|(\mathbf{I}_{n+2-\beta,4R}f_j)^\sigma\|_{\frac{p_1}{\sigma}}^\lambda \\ &\leq C\|f_j\|_p^{\lambda\sigma}, \end{aligned}$$

by (6.29). It follows therefore from (6.26) that

$$\|r_j^\delta f_j\|_{L^q(B_R(0))} \leq \frac{C}{R^\alpha} \left[1 + \|r_j^\gamma f_j\|_{L^p(B_{4R}(0))}^{\lambda\sigma} \right],$$

which is a bounded sequence by (6.22).

It remains to prove that q satisfies (6.24) for some positive constant $C_0 = C_0(n, \lambda, \sigma, \alpha, \beta)$. By (6.28) and (6.32) we have

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &= \frac{1}{p} - \frac{\lambda}{p_2} = \frac{1}{p} - \lambda \left[\frac{\sigma}{p_1} - \frac{n+2-\alpha-\varepsilon}{n} \right] \\ &= \frac{1}{p} + \frac{\lambda(n+2-\alpha-\varepsilon)}{n} - \frac{\lambda\sigma}{p_1} \\ &= \frac{1}{p} + \frac{\lambda(n+2-\alpha-\varepsilon)}{n} - \lambda\sigma \left[\frac{1}{p} - \frac{n+2-\beta-\varepsilon}{n} \right] \\ &= \frac{1-\lambda\sigma}{p} + \frac{\lambda(n+2-\alpha-\varepsilon) + \lambda\sigma(n+2-\beta-\varepsilon)}{n}. \end{aligned}$$

If $\lambda\sigma \leq 1$ then

$$\frac{1}{p} - \frac{1}{q} \geq \frac{\lambda(n+2-\alpha-\varepsilon) + \lambda\sigma(n+2-\beta-\varepsilon)}{n} = C_1(n, \lambda, \sigma, \alpha, \beta) > 0.$$

by (6.25) and (6.7). If $\lambda\sigma > 1$ then

$$\begin{aligned} \frac{1}{p} - \frac{1}{q} &\geq 1 - \lambda\sigma + \frac{\lambda(n+2-\alpha-\varepsilon) + \lambda\sigma(n+2-\beta-\varepsilon)}{n} \\ &= \frac{\lambda(\beta-2+\varepsilon)}{n} \left[\frac{n + \lambda(n+2-\alpha-\varepsilon)}{\lambda(\beta-2+\varepsilon)} - \sigma \right] \\ &= C_2(n, \lambda, \sigma, \alpha, \beta) > 0 \end{aligned}$$

by (6.25) and (6.7). Thus (6.24) holds with $C_0 = \min\{C_1, C_2\}$. This completes the proof of Lemma 6.2. \square

We are now ready to complete the proof of (6.9). By (6.14), the sequence $\{f_j\}$ is bounded in $L^1(B_2(0))$. Starting with this fact and iterating Lemma 6.2 a finite number of times (m times is enough if $m > 1/C_0$) we see that there exists $R_0 \in (0, \frac{1}{2})$ and $\gamma > n$ such that sequence $\{r_j^\gamma f_j\}$ is bounded in $L^\infty(B_{R_0}(0))$. In particular $\{r_j^\gamma f_j(0)\}$ is a bounded sequence, whence (6.11) and (6.9).

Step 2: Proof of (6.2).

Let $\{x_j\} \subset \mathbb{R}^n$ be a sequence satisfying (6.10). Then, as is Step 1, f and g satisfy (6.12) and (6.13) where $r_j = |x_j|/4$.

By (6.9), for some $\gamma > n$, we have

$$f(x) \leq C|x_j|^{-\gamma} \quad \text{for } |x - x_j| < 2r_j. \quad (6.33)$$

Let

$$\hat{\mathbf{I}}_{a,j} f(x) = \int_{|y-x_j| < 2r_j} \frac{f(y) dy}{|x-y|^{n-a}} \quad \text{and} \quad \hat{\mathbf{U}}_{a,b,\sigma,j} f = \hat{\mathbf{I}}_{a,j}((\hat{\mathbf{I}}_{b,j} f)^\sigma).$$

Since, by (6.13),

$$g(x) \leq C \left[|x_j|^{-\sigma(\beta-2)} + \left(\hat{\mathbf{I}}_{n+2-\beta,j} f \right)^\sigma(x) \right] \quad \text{for } |x - x_j| < r_j, \quad (6.34)$$

we find that

$$\begin{aligned}
\int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{\alpha-2}} &\leq C \left[\int_{|y-x_j|<r_j} \frac{|x_j|^{-\sigma(\beta-2)}}{|x_j - y|^{\alpha-2}} dy + \int_{|y-x_j|<2r_j} \frac{(\hat{\mathbf{I}}_{n+2-\beta,j} f(y))^\sigma}{|x_j - y|^{\alpha-2}} dy \right] \\
&\leq C \left[|x_j|^{n+2-\alpha-\sigma(\beta-2)} + \|\hat{\mathbf{U}}_{n+2-\alpha,n+2-\beta,\sigma,j} f\|_{L^\infty(B_{2r_j}(x_j))} \right] \\
&= \mathcal{O}(|x_j|^{-(\alpha-2)}) + o\left(|x_j|^{-\frac{\gamma}{n}[\sigma(\beta-2)-(n+2-\alpha)]}\right) \quad \text{as } j \rightarrow \infty \quad (6.35)
\end{aligned}$$

where the big “oh” term follows from (6.1) and the little “oh” term follows from (6.33), $f \in L^1(B_1(0))$, (6.8), and Proposition 3.1.

Since $g \in L^1(B_1(0))$ we have

$$\begin{aligned}
\int_{|y-x_j|<2r_j} \frac{g(y) dy}{|x_j - y|^{\alpha-2}} &\leq \int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{\alpha-2}} + \int_{r_j<|y-x_j|<2r_j} \frac{g(y)}{r_j^{\alpha-2}} dy \\
&\leq \int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{\alpha-2}} + o(|x_j|^{2-\alpha}) \quad \text{as } j \rightarrow \infty.
\end{aligned}$$

We therefore deduce from (6.12) and (6.35) that

$$f(x_j) = \mathcal{O}(|x_j|^{-\lambda(\alpha-2)}) + o\left(|x_j|^{-\frac{\lambda\gamma}{n}[\sigma(\beta-2)-(n+2-\alpha)]}\right) \quad \text{as } j \rightarrow \infty.$$

Thus, since $\{x_j\}$ was an arbitrary sequence satisfying (6.10), we have

$$f(x) = \mathcal{O}(|x|^{-\lambda(\alpha-2)}) + o\left(|x|^{-\frac{\lambda\gamma}{n}[\sigma(\beta-2)-(n+2-\alpha)]}\right) \quad \text{as } x \rightarrow 0. \quad (6.36)$$

Let $\{\gamma_j\}$ be a sequence of real numbers defined by $\gamma_0 = \gamma$ and

$$\gamma_{j+1} = \frac{\lambda\gamma_j}{n}[\sigma(\beta-2) - (n+2-\alpha)] \quad \text{for } j = 0, 1, \dots$$

Since σ satisfies (6.1) and (6.8) we have $\{\gamma_j\} \subset (0, \infty)$ and $\gamma_j \rightarrow 0$ as $j \rightarrow \infty$. Thus, iterating finitely many times the procedure of going from (6.9) to (6.36) we obtain (6.2).

We now prove (6.5). Since increasing σ weakens the conditions on f and g in the system (1.6) and since increasing σ to a value slightly larger than $(n+2-s)/(\beta-2)$ does not change the estimate (6.5), we can assume for the proof of (6.5) that

$$\sigma > \frac{n+2-s}{\beta-2}. \quad (6.37)$$

Let $\{x_j\} \subset \mathbb{R}^n$ be a sequence satisfying (6.10). Then, as before, g satisfies (6.34) where $r_j = |x_j|/4$. Repeating the calculation (6.35), except this time with $\alpha = s$ and $\gamma = \lambda(\alpha-2)$ and using (6.37) instead of (6.8), we get

$$\int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{s-2}} = \mathcal{O}(|x_j|^{2-s}) + o\left(|x_j|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-(n+2-s)]}{n}}\right) \quad \text{as } j \rightarrow \infty.$$

Thus

$$\begin{aligned}
\int_{|y|<1} \frac{g(y) dy}{|x_j - y|^{s-2}} &= \int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{s-2}} + \int_{|y-x_j|>r_j, |y|<1} \frac{g(y) dy}{|x_j - y|^{s-2}} \\
&\leq C|x_j|^{2-s} + \int_{|y-x_j|<r_j} \frac{g(y) dy}{|x_j - y|^{s-2}} \\
&= \mathcal{O}(|x_j|^{2-s}) + o\left(|x_j|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-(n+2-s)]}{n}}\right) \quad \text{as } j \rightarrow \infty
\end{aligned}$$

which proves (6.5). This finishes the proof of Theorem 6.1. \square

We are now able to easily prove Theorems 2.3–2.5.

Proof of Theorems 2.3–2.5. By Lemma 3.9, u and v satisfy (3.17)–(3.20). Let $f = -\Delta u$ and $g = -\Delta v$. By (3.17), (3.20), and Corollary 3.7, f and g are $L^1(B_1(0))$ solutions of (1.6) for some positive constant M . Hence, by Theorem 6.1, f and g satisfy (6.4) and (6.5) with $s = n$. It therefore follows from (3.19) that

$$\begin{aligned}
u(x) &= \mathcal{O}\left(|x|^{-(n-2)}\right) + o\left(|x|^{-\frac{\lambda(\alpha-2)(n-2)}{n}}\right) \quad \text{as } x \rightarrow 0 \\
v(x) &= \mathcal{O}\left(|x|^{-(n-2)}\right) + o\left(|x|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-2]}{n}}\right) \quad \text{as } x \rightarrow 0
\end{aligned}$$

which immediately gives Theorems 2.3–2.5. \square

7 Proof of Theorem 2.6

Define continuous functions $\varphi, \psi : (0, 1) \rightarrow (0, 1)$ by

$$\varphi = \max\{h^{1/2}, h^{1/(2\sigma)}\} \quad \text{and} \quad \psi(t) = B\varphi(t)\sigma t^{\frac{\lambda(\alpha-2)}{n}(n-\sigma(\beta-2))} \quad (7.1)$$

where $B = B(n, \beta, \sigma)$ is a positive constant to be specified later.

Let $\{x_j\} \subset \mathbb{R}^n$ be a sequence satisfying

$$0 < 4|x_{j+1}| < |x_j| < 1/2, \quad \sum_{j=1}^{\infty} \varphi(|x_j|) < \infty, \quad \text{and} \quad \sum_{j=1}^{\infty} \psi(|x_j|) < \infty,$$

and let $r_j = |x_j|^{\lambda(\alpha-2)/n}$. Then by Lemma 3.3 there exist a positive constant $A = A(n)$ and positive functions $u, \hat{v} \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that (3.8)–(3.11) hold as stated and also with u and φ replaced with \hat{v} and ψ respectively. Let

$$v = \hat{v} + |x|^{-(n-2)}.$$

As in the proof of Theorem 2.2, we can modify u and v on $\mathbb{R}^n \setminus B_2(0)$ in such a way that they become $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ functions, and, by (3.9), u and v will satisfy (1.1) in $B_2(0) \setminus \{0\}$ provided they satisfy (1.1) in $\cup_{j=1}^{\infty} B_{r_j}(x_j)$.

Since, as the proof of Lemma 3.6 shows,

$$\int_{|y|<2} \frac{1}{|x-y|^\alpha} \frac{1}{|y|^{n-2}} dy \geq \frac{C}{|x|^{\alpha-2}} \quad \text{for } 0 < |x| < 2,$$

we have for $|x - x_j| < r_j$ that

$$\begin{aligned} \left(\frac{1}{|x|^\alpha} * v \right)^\lambda &\geq \left(\int_{|y|<2} \frac{1}{|x-y|^\alpha} \frac{1}{|y|^{n-2}} dy \right)^\lambda \\ &\geq \frac{C}{|x_j|^{\lambda(\alpha-2)}} > \frac{\varphi(|x_j|)}{r_j^n} \geq -\Delta u. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{u(x_j)}{h(|x_j|)|x_j|^{-\lambda(\alpha-2)(n-2)/n}} &\geq \frac{A\varphi(|x_j|)r_j^{-(n-2)}}{\varphi(|x_j|)^2 r_j^{-(n-2)}} \\ &= \frac{A}{\varphi(|x_j|)} \rightarrow \infty \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Also, for $|x - x_j| < r_j$,

$$\begin{aligned} \left(\frac{1}{|x|^\beta} * u \right)^\sigma &\geq \left(\int_{|y-x_j|<r_j} \frac{u(y)}{|x-y|^\beta} dy \right)^\sigma \\ &\geq \left(\frac{A\varphi(|x_j|)|B_{r_j}(x_j)|}{r_j^{n-2}(2r_j)^\beta} \right)^\sigma = B\varphi(|x_j|)^\sigma r_j^{-(\beta-2)\sigma} \end{aligned}$$

where $B = (2^{-\beta}|B_1(0)|A)^\sigma$. Hence (7.1) implies

$$\begin{aligned} \left(\frac{1}{|x|^\beta} * u \right)^\sigma &\geq \psi(|x_j|)r_j^{-(n-\sigma(\beta-2))} r_j^{-(\beta-2)\sigma} \\ &= \psi(|x_j|)r_j^{-n} \geq -\Delta \hat{v} = -\Delta v \quad \text{for } |x - x_j| < r_j. \end{aligned}$$

Finally, again by (7.1),

$$\begin{aligned} \frac{v(x_j)}{h(|x_j|)|x_j|^{-\frac{\lambda(\alpha-2)[\sigma(\beta-2)-2]}{n}}} &\geq \frac{A\psi(|x_j|)r_j^{-(n-2)}}{\varphi(|x_j|)^{2\sigma} r_j^{-(\sigma(\beta-2)-2)}} \\ &= \frac{AB}{\varphi(|x_j|)^\sigma} \rightarrow \infty \quad \text{as } j \rightarrow \infty \end{aligned}$$

and

$$\frac{v(x_j)}{h(|x_j|)|x_j|^{-(n-2)}} \geq \frac{1}{h(|x_j|)} \rightarrow \infty \quad \text{as } j \rightarrow \infty.$$

This completes the proof of Theorem 2.6.

8 Proof of Theorem 2.7

Choose $M > 1$ such that $\int_{|y|<2} M|x-y|^{-\beta} dy > 1$ for $|x| < 2$. The positive functions u and v that we construct will satisfy not just (1.1), (2.9), and (2.10) but also

$$u \geq M \quad \text{in } B_2(0) \setminus \{0\}. \quad (8.1)$$

If u and v are positive functions satisfying (1.1) and (8.1) then u and v also satisfy (1.1) for any larger value of σ because then

$$\int_{\mathbb{R}^n} \frac{u(y) dy}{|x-y|^\beta} \geq \int_{|y|<2} \frac{M dy}{|x-y|^\beta} > 1 \quad \text{for } 0 < |x| < 2.$$

Hence we can assume for the proof of Theorem 2.7 that

$$\sigma < \frac{n}{\beta-2}. \quad (8.2)$$

Define

$$a = \frac{1}{\lambda(\alpha-2)-n} \quad \text{and} \quad b = \frac{1}{n-\sigma(\beta-2)}.$$

Using (2.8) and (8.2) we have

$$a, b > 0 \quad \text{and} \quad a\lambda - b < 0. \quad (8.3)$$

Let $\varphi : (0, 1) \rightarrow (0, 1)$, $\{x_j\} \subset \mathbb{R}^n$, $\{r_j\} \subset (0, 1)$ and $A = A(n)$ be as in Lemma 3.3. By Lemma 3.3 there exists a positive function $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ that satisfies (3.8)–(3.11).

Since adding a positive constant to u will not change the fact that u satisfies (3.8)–(3.11), we can assume, instead of (3.10), that $u > M$ in $B_2(0) \setminus \{0\}$ where M is as stated above. As in the proof of Theorem 2.2, we modify u on $\mathbb{R}^n \setminus B_2(0)$ in such a way as to obtain a $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ function. For every $j \geq 1$ we define ψ_j as a function of r_j by

$$r_j = \left[\frac{(B\psi_j)^\lambda}{\varphi(|x_j|)} \right]^a, \quad \text{where} \quad B = B(n) = \frac{A|B_1(0)|}{2^n} > 0. \quad (8.4)$$

Note that

$$\frac{A\psi_j}{r_j^{n-2}} = \frac{A}{B^{a\lambda(n-2)}} \frac{\varphi(|x_j|)^{a(n-2)}}{\psi_j^{\lambda a(n-2)-1}}.$$

By decreasing r_j (and thereby decreasing ψ_j) we may assume

$$\sum_{j=1}^{\infty} \psi_j < \infty,$$

$$\psi_j^{a\lambda-b} \geq \frac{\varphi(|x_j|)^{a-b\sigma}}{B^{a\lambda+b\sigma}}, \quad (8.5)$$

and

$$\frac{A\varphi(|x_j|)}{r_j^{n-2}} \gg h(|x_j|), \quad \frac{A\psi_j}{r_j^{n-2}} \gg h(|x_j|) \quad \text{as } j \rightarrow \infty. \quad (8.6)$$

It follows from (8.5) and (8.4) that

$$\left(\frac{B\varphi(|x_j|)}{r_j^{\beta-2}} \right)^\sigma \geq \frac{\psi_j}{r_j^n}. \quad (8.7)$$

Let $\psi : (0, 1) \rightarrow (0, 1)$ be a continuous function such that $\psi(|x_j|) = \psi_j$. By Lemma 3.3 there exists a positive function $v \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that

$$0 \leq -\Delta v \leq \frac{\psi(|x_j|)}{r_j^n} \quad \text{in } B_{r_j}(x_j), \quad (8.8)$$

$$-\Delta v = 0 \quad \text{in } \mathbb{R}^n \setminus \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \right), \quad (8.9)$$

$$v \geq 1 \quad \text{in } \mathbb{R}^n \setminus \{0\}, \quad (8.10)$$

$$v \geq \frac{A\psi(|x_j|)}{r_j^{n-2}} \quad \text{in } B_{r_j}(x_j). \quad (8.11)$$

We modify v on $\mathbb{R}^n \setminus B_2(0)$ in such a way as to obtain a $C^\infty(\mathbb{R}^n \setminus \{0\}) \cap L^1(\mathbb{R}^n)$ function. In order to check that u and v satisfy (1.1) let us remark first that by (8.11) we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{v(y) dy}{|x-y|^\alpha} &\geq \frac{A\psi(|x_j|)}{r_j^{n-2}} \int_{B_{r_j}(x_j)} \frac{dy}{|x-y|^\alpha} \\ &\geq \frac{A\psi(|x_j|)}{r_j^{n-2}} \frac{|B_{r_j}(x_j)|}{(2r_j)^\alpha} \geq \frac{B\psi(|x_j|)}{r_j^{\alpha-2}} \quad \text{for } x \in B_{r_j}(x_j) \end{aligned} \quad (8.12)$$

and similarly

$$\int_{\mathbb{R}^n} \frac{u(y) dy}{|x-y|^\beta} \geq \frac{B\varphi(|x_j|)}{r_j^{\beta-2}} \quad \text{for } x \in B_{r_j}(x_j). \quad (8.13)$$

Now, by (3.8), (8.4), (8.7), (8.8), (8.12) and (8.13) we deduce that u and v are solutions of (1.1). Finally, to check that u and v satisfy (2.9) and (2.10) along the sequence $\{x_j\}$ we use (3.11), (8.11) and (8.6).

9 Proof of Theorem 2.11

We consider two cases.

Case I. Suppose $\lambda(\alpha - 2) < n$. Let $\chi : \mathbb{R}^n \rightarrow [0, 1]$ be a C^∞ function such that $\chi = 1$ for $|x| < 2\varepsilon$ and $\chi = 0$ for $|x| > 4\varepsilon$. Then $f(x) := \varepsilon|x|^{-\lambda(\alpha-2)}\chi(x)$ and $g(x) := \varepsilon|x|^{-\sigma(\beta-2)}\chi(x)$ clearly satisfy (2.12)–(2.15).

Case II. Suppose $\lambda(\alpha - 2) \geq n$. Define $\varphi : (0, 1) \rightarrow (0, 1)$ by $\varphi = h^{n/(2\sigma(n+2-\beta))}$. Let $\{x_j\}$ be a sequence in \mathbb{R}^n such that

$$0 < 4|x_{j+1}| < |x_j| < \varepsilon/2.$$

Let

$$\varepsilon_j = \varphi(|x_j|) \quad \text{and} \quad r_j = \left(\frac{\varepsilon_j}{\varepsilon} \right)^{1/n} (2|x_j|)^{\lambda(\alpha-2)/n}. \quad (9.1)$$

By taking a subsequence we can assume

$$0 < r_j < |x_j|/2 < 1 \quad \text{and} \quad \varepsilon_j < 2^{-j}. \quad (9.2)$$

Thus

$$\sum_{j=1}^{\infty} \varepsilon_j < \infty. \quad (9.3)$$

Let

$$\delta_j = \varepsilon J^\sigma \varepsilon_j^\sigma r_j^{n-\sigma(\beta-2)} \quad (9.4)$$

where $J = J(n) > 0$ is a constant to be specified later. By (9.2), $\delta_j \leq \varepsilon J^\sigma 2^{-\sigma j}$ and hence

$$\sum_{j=1}^{\infty} \delta_j < \infty. \quad (9.5)$$

Define sequences $\{M_j\}$ and $\{N_j\}$ by

$$M_j = \frac{\varepsilon_j}{r_j^n} \quad \text{and} \quad N_j = \frac{\delta_j}{r_j^n}. \quad (9.6)$$

Then by (9.1) and (9.2),

$$M_j = \frac{\varepsilon}{(2|x_j|)^{\lambda(\alpha-2)}} < \frac{\varepsilon}{|x|^{\lambda(\alpha-2)}} \quad \text{for } |x - x_j| < r_j. \quad (9.7)$$

Let $\psi : \mathbb{R}^n \rightarrow [0, 1]$ be a C^∞ function such that $\psi = 0$ in $\mathbb{R}^n \setminus B_1(0)$ and $\psi(0) = 1$. Define $\psi_j : \mathbb{R}^n \rightarrow [0, 1]$ by $\psi_j(y) = \psi(\eta)$ where $y = x_j + r_j \eta$. Then

$$\psi_j(x_j) = 1 \quad \text{and} \quad \int_{\mathbb{R}^n} \psi_j(y) dy = \int_{\mathbb{R}^n} \psi(\eta) r_j^n d\eta = r_j^n I \quad (9.8)$$

where $I = \int_{\mathbb{R}^n} \psi(\eta) d\eta > 0$.

Define $f, g : \mathbb{R}^n \setminus \{0\} \rightarrow [0, \infty)$ by

$$f = \sum_{j=1}^{\infty} M_j \psi_j \quad \text{and} \quad g = \sum_{j=1}^{\infty} N_j \psi_j.$$

Since the functions ψ_j have disjoint supports, $f, g \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and by (9.8) and (9.6) we have

$$\int_{\mathbb{R}^n} f(y) dy = I \sum_{j=1}^{\infty} \varepsilon_j \quad \text{and} \quad \int_{\mathbb{R}^n} g(y) dy = I \sum_{j=1}^{\infty} \delta_j.$$

Thus, by (9.3) and (9.5), we see that $f, g \in L^1(\mathbb{R}^n)$.

From (9.8) and (9.7) we have

$$|x_j|^{\lambda(\alpha-2)} f(x_j) = M_j |x_j|^{\lambda(\alpha-2)} = \frac{\varepsilon}{2^{\lambda(\alpha-2)}}$$

and

$$|x|^{\lambda(\alpha-2)} f(x) \leq M_j |x|^{\lambda(\alpha-2)} < \varepsilon \quad \text{for } |x - x_j| < r_j.$$

Thus f satisfies (2.14) and the first line of (2.13). (Note that we only need to check (2.13) holds in $\cup_{j=1}^{\infty} B_{r_j}(x_j)$ because elsewhere $f = g = 0$.)

For $x = x_j + r_j \xi$ and $|\xi| < 1$ we have

$$\begin{aligned}
\left(\int_{|y| < \varepsilon} \frac{f(y) dy}{|x - y|^{\beta-2}} \right)^\sigma &\geq \left(\int_{|y - x_j| < r_j} \frac{M_j \psi_j(y) dy}{|x - y|^{\beta-2}} \right)^\sigma = \left(\int_{|\eta| < 1} \frac{M_j \psi(\eta) r_j^n d\eta}{r_j^{\beta-2} |\xi - \eta|^{\beta-2}} \right)^\sigma \\
&= \left(\frac{\varepsilon_j}{r_j^{\beta-2}} \int_{|\eta| < 1} \frac{\psi(\eta) d\eta}{|\xi - \eta|^{\beta-2}} \right)^\sigma \\
&\geq \left(\frac{J \varepsilon_j}{r_j^{\beta-2}} \right)^\sigma \quad \text{where } J = \min_{|\xi| \leq 1} \int_{|\eta| < 1} \frac{\psi(\eta) d\eta}{|\xi - \eta|^{\beta-2}} > 0 \\
&= \frac{1}{\varepsilon} \frac{\delta_j}{r_j^n} = \frac{1}{\varepsilon} N_j \geq \frac{1}{\varepsilon} g(x)
\end{aligned}$$

by (9.4). Thus the second line of (2.13) holds.

Finally, by (9.4) and (9.1),

$$\begin{aligned}
g(x_j) = N_j = \frac{\delta_j}{r_j^n} &= \frac{C \varepsilon_j^\sigma}{r_j^{\sigma(\beta-2)}} \\
&= \frac{C \varepsilon_j^\sigma}{\varepsilon_j^{\sigma(\beta-2)/n} |x_j|^{\lambda(\alpha-2)\sigma(\beta-2)/n}} \\
&= \frac{C \varepsilon_j^{\sigma(n+2-\beta)/n}}{|x_j|^{\lambda(\alpha-2)\sigma(\beta-2)/n}} = \frac{C \sqrt{h(|x_j|)}}{|x_j|^{\lambda(\alpha-2)\sigma(\beta-2)/n}}
\end{aligned}$$

which gives (2.15).

10 Proof of Theorem 2.12

The functions f and g that we construct will satisfy not just (2.17) and (2.13) but also

$$f = g = 0 \quad \text{in } \mathbb{R}^n \setminus B_\varepsilon(0). \quad (10.1)$$

If f and g satisfy (2.17), (2.13), and (10.1) then they also satisfy (2.13) for any larger value of σ . Hence we can assume without loss of generality that

$$\sigma < \frac{n}{\beta - 2}. \quad (10.2)$$

Let $\{x_j\} \subset \mathbb{R}^n$ and $\{\varepsilon_j\} \subset (0, 1)$ be sequences in such that

$$0 < 4|x_{j+1}| < |x_j| < \varepsilon/2$$

and

$$\sum_{j=1}^{\infty} \varepsilon_j < \infty. \quad (10.3)$$

Choose $r_j \in (0, |x_j|/2)$ such that

$$N_j := \varepsilon \left(\frac{J\varepsilon_j}{r_j^{\beta-2}} \right)^\sigma \geq h(|x_j|)^2, \quad M_j := \frac{\varepsilon_j}{r_j^n} \geq h(|x_j|)^2,$$

$$\delta_j := \varepsilon J^\sigma \varepsilon_j^\sigma r_j^{n-\sigma(\beta-2)} < 2^{-j},$$

and

$$r_j^{\lambda(\beta-2)[\sigma - (\frac{n+2-\alpha}{\beta-2} + \frac{n-1}{\beta-2} \frac{1}{\lambda})]} \leq \varepsilon^{\lambda+1} J^{\lambda(\sigma+1)} \varepsilon_j^{\lambda\sigma-1}$$

where $J = J(n) > 0$ is a constant to specified later. (This is possible because the exponents on r_j in all of these conditions are positive by (2.16) and (10.2).) Then

$$h(|x_j|)^2 \leq N_j = \frac{\delta_j}{r_j^n} = \varepsilon \left(\frac{J\varepsilon_j}{r_j^{\beta-2}} \right)^\sigma, \quad (10.4)$$

$$h(|x_j|)^2 \leq M_j = \frac{\varepsilon_j}{r_j^n} \leq \varepsilon \left(\frac{J\delta_j}{r_j^{\alpha-2}} \right)^\lambda, \quad (10.5)$$

and

$$\sum_{j=1}^{\infty} \delta_j < \infty. \quad (10.6)$$

Define ψ , ψ_j , I , J , f , and g as in the proof of Theorem 2.11. Then f and g satisfy (10.1). Moreover, using (10.3)–(10.6), we see as in the proof of Theorem 2.11 that (2.17) holds,

$$\left(\int_{|y|<\varepsilon} \frac{f(y) dy}{|x-y|^{\beta-2}} \right)^\sigma \geq \frac{1}{\varepsilon} g(x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}$$

and

$$\left(\int_{|y|<\varepsilon} \frac{g(y) dy}{|x-y|^{\alpha-2}} \right)^\lambda \geq \frac{1}{\varepsilon} f(x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

Thus f and g satisfy (2.13). Also

$$f(x_j) = M_j \geq h(|x_j|)^2 \gg h(|x_j|) \quad \text{as } j \rightarrow \infty$$

and

$$g(x_j) = N_j \geq h(|x_j|)^2 \gg h(|x_j|) \quad \text{as } j \rightarrow \infty.$$

Hence (2.18) and (2.19) hold.

Acknowledgement. The authors would like to thank Stephen J. Gardiner for helpful discussions.

References

- [1] M.-F. Bidaut-Véron and Th. Raoux, Asymptotics of solutions of some nonlinear elliptic systems, *Comm. Partial Differential Equations* 21 (1996), 1035–1086.
- [2] H. Brezis and P.-L. Lions, A note on isolated singularities for linear elliptic equations, *Mathematical analysis and applications, Part A*, pp. 263-266, *Adv. in Math. Suppl. Stud.*, 7a, Academic Press, New York-London, 1981.
- [3] G. Caristi, L. D’Ambrosio and E. Mitidieri, Representation formulae for solutions to some classes of higher order systems and related Liouville theorems, *Milan J. Math.* 76 (2008), 27-67.
- [4] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, *Comm. Partial Differential Equations* 30 (2005), 59-65.
- [5] W. Chen and C. Li, An integral system and the Lane-Emden conjecture, *Discrete Contin. Dyn. Syst.* 24 (2009), 1167-1184.
- [6] J. T. Devreese and A. S. Alexandrov, *Advances in polaron physics*, Springer Series in Solid-State Sciences, vol. 159, Springer, 2010.
- [7] M. Ghergu, A. Moradifam and S.D. Taliaferro, Isolated singularities of polyharmonic inequalities, *J. Functional Anal.* 261 (2011), 660-680.
- [8] M. Ghergu, S.D. Taliaferro and I.E. Verbitsky, Pointwise bounds and blow-up for systems of semilinear elliptic inequalities at an isolated singularity via nonlinear potential estimates, <http://arxiv.org/abs/1402.0113>
- [9] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd edition, Springer-Verlag, Berlin, 1983.
- [10] V.P. Havin and V.G. Maz’ya, Nonlinear potential theory, *Usp. Mat. Nauk* 27 (1972), 67-138 (in Russian). English translation: *Russ. Math. Surv.* 27 (1972), 71-148.
- [11] L.I. Hedberg, On certain convolution inequalities, *Proc. Amer. Math. Soc.* 36 (1972), 505-510.
- [12] C. Jin and C. Li, Quantitative analysis of some system of integral equations, *Calc. Var. Partial Differential Equations* 26 (2006), 447-457.
- [13] K.R.W. Jones, Newtonian quantum gravity, *Australian Journal of Physics* 48 (1995), 1055-1081.
- [14] Y. Lei, On the regularity of positive solutions of a class of Choquard type equations, *Math. Z.* 273 (2013), 883-905.
- [15] Y. Lei, Qualitative analysis for the static Hartree-type equations, *SIAM J. Math. Anal.* 45 (2013), 388-406.
- [16] Y. Lei, C. Li and C. Ma, Asymptotic radial symmetry and growth estimates of positive solutions to weighted Hardy-Littlewood-Sobolev system of integral equations, *Calc. Var. Partial Differential Equations* 45 (2012), 43-61.

- [17] E.H. Lieb, Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation, *Studies in Appl. Math.* 57 (1976/77), 93-105.
- [18] I.M. Moroz, R. Penrose and P. Tod, Spherically-symmetric solutions of the Schrödinger-Newton equations, *Classical Quantum Gravity* 15 (1998), 2733-2742.
- [19] S. Pekar, *Untersuchung über die Elektronentheorie der Kristalle*, Akademie Verlag, Berlin, 1954.
- [20] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [21] Ph. Souplet, The proof of the Lane-Emden conjecture in four space dimensions, *Adv. Math.* 221 (2009), 1409–1427.
- [22] S.D. Taliaferro, Isolated singularities of nonlinear parabolic inequalities, *Math. Ann.* 338 (2007), 555-586.
- [23] S.D. Taliaferro, Initial blow-up of solutions of semilinear parabolic inequalities, *J. Differential Equations* 250 (2011), 892-928.